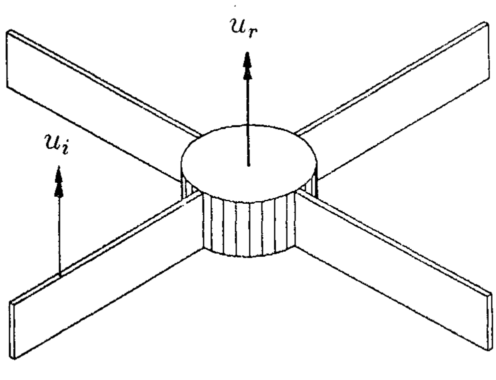

Fig. 1 Flexible spacecraft with controllers.

spacecraft to the rest state and to suppress the bending vibration of the appendages. A single-axis maneuver is considered in this example for simplicity.

Let the candidate Lyapunov function be made up of the total energy and a pseudopotential energy of rotation:

$$L = E + \tfrac{1}{2} k_\theta (\theta - \theta_f)^2, \qquad k_\theta > 0 \tag{26}$$

where θ represents the present rotational angle and θ_f is the object angle (fixed). From the Work-Energy Rate Principle,

$$\dot{E} = u_r \dot{\theta} + \sum_{i=1}^{n} u_i \left[\dot{\theta} + \frac{\partial}{\partial t} \left(\frac{\partial y_i}{\partial x} \right) \right] \tag{27}$$

where u_r is the central controller torque and u_i is the ith appendage controller torque. Let $\dot{\theta}_i \equiv (\partial/\partial t)[(\partial y_i/\partial x)]$ be the rate of change of slope at the ith appendage controller location. Then, the time derivative of L is

$$\dot{L} = u_r \dot{\theta} + k_\theta (\theta - \theta_f) \dot{\theta} + \sum u_i (\dot{\theta} + \dot{\theta}_i) \tag{28}$$

The hub and appendage control laws can then be obtained as

$$u_r = -K_r \dot{\theta} - k_\theta (\theta - \theta_f) \tag{29}$$

$$u_i = -K_i (\dot{\theta} + \dot{\theta}_i) \tag{30}$$

where k_θ, K_r, and K_i are positive gains. Note that this approach does not need the assumption of small deflections of the appendages and consideration of specific mode shapes. All controllable modes are stabilized and no discretization errors corrupt the stability arguments. Derivations of control laws for the system from the hybrid ordinary/partial differential equations of motion (without using the Work-Energy Rate Principle) can be found in Refs. 4–6.

Conclusions

The Work-Energy Rate Principle in the generalized coordinate space is applied to design feedback control laws for the class of systems that admit total energy as a part of the Lyapunov function. It is shown that the use of this principle reduces the effort required to design feedback control laws for scleronomic and holonomic systems. It is also shown that equations of motion need not be derived to design the feedback control laws and all developments are extensions of the work/energy method and the Principle of Virtual Work. All of the feedback control laws derived in these examples render the closed-loop system asymptotically stable. This method can be applied to rheonomic or nonholonomic systems by properly defining the constraint forces.

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Kane's Equations, Lagrange's Equations, and Virtual Work

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Introduction

THE literature does not appear to contain a derivation of Kane's equations from some first principle. In general, Kane's generalized forces are presented as definitions, from which Kane's equations then follow.[1-4] In this Note, Kane's generalized forces and equations are derived from a first principle—the work-energy form of Newton's second law. Lagrange's equations can also be derived from this same basic form; although it differs conceptually from the usual virtual work (also known as d'Alembert principle) derivations, many of the steps are similar. These parallel derivations clearly show the commonality of Kane's and Lagrange's equations and the occurrence of virtual work-type terms.

The common feature of Lagrange's and Kane's equations is the transformation to generalized coordinates so that system constraint forces are (or can be) eliminated. In the usual derivations of Lagrange's equations, this is attributed to the restricted character of the virtual displacements[5-13]: displacements "for which the virtual work of the forces of constraint vanishes."[5] In Kane's equations, forces of constraint are eliminated if the vector multiplied (dot product) into Newton's law is chosen properly.[2] In some derivations of Lagrange's equations, the vector multiplied into Newton's law is similarly chosen.[11,14] In all cases, explicit time-varying (rheonomic) kinematical terms are discarded, or simply not considered, often without comment. When explanations of virtual displacements and properly chosen vectors are presented, they tend to be convoluted and unenlightening.[2,5-13] The present derivations avoid these ambiguities and explanations and possibly provide increased generality.

Derivations

The derivations are for a holonomic system of p constant-mass particles. Inclusion of rigid bodies is straightforward.

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The presence of nonholonomic constraints does not affect these results; their incorporation is discussed in the respective sections.

Newton's second law for a particle i of mass m_i located at r^i in an inertial reference frame is

$$F_i = m_i \ddot{r}^i = m_i a^i, \qquad i = 1,p \tag{1}$$

where F_i is the sum of all forces acting on the particle: the known applied and external forces plus the constraint forces necessary to maintain the configuration. The constraint forces are usually unknown and must be eliminated to solve for the system motions.

The total work of all forces on the p particles in going through some possible displacements dr^i, $i = 1,p$, is

$$dW = \sum_{i=1}^{p} F_i \cdot dr^i = \sum_{i=1}^{p} m_i a^i \cdot dr^i \tag{2}$$

To incorporate the geometric (holonomic) constraints, the $3p$ coordinates r^i are transformed to n generalized coordinates q_j, viz.,

$$r^i = r^i(q_1, q_2, \ldots, q_n, t), \qquad i = 1,p \tag{3}$$

The usual and most important cases are for $n \ll 3p$. The possible differential displacements dr^i are given by

$$dr^i = \sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j} dq_j + \frac{\partial r^i}{\partial t} dt, \qquad i = 1,p \tag{4}$$

The corresponding possible absolute velocities of the m_i are

$$v^i = \frac{dr^i}{dt} = \dot{r}^i = \sum_{j=1}^{n} \frac{\partial r^i}{\partial q_j} \dot{q}_j + \frac{\partial r^i}{\partial t}, \qquad i = 1,p \tag{5}$$

The quantities in Eqs. (4) and (5) are ordinary differentials and derivatives, occurring with the passage of time dt.

Kane's Equations

Kane and Levinson observe that Newton's law multiplied (dot product) by any conformable vector remains valid, but to get useful generalized forces, the vector must be chosen properly.[2] Their vector resembles a virtual displacement, a concept that they dismiss. In the following, Kane's generalized forces and the appropriate vector are explicitly derived.

Accompanying the n generalized coordinates q_j are n so-called generalized speeds u_r. The u_r are linear functions of the $\dot{q}_j$ and are defined by the following reciprocal relationships:

$$u_r = \sum_{j=1}^{n} Y_{rj} \dot{q}_j + Z_r, \qquad r = 1,n \tag{6}$$

$$\dot{q}_j = \sum_{r=1}^{n} W_{jr} u_r + X_j, \qquad j = 1,n \tag{7}$$

Y_{rj} and Z_r, and W_{jr} and X_j may be functions of the q_j and t. The u_r are sometimes called nonholonomic velocities, time derivatives of quasicoordinates, quasivelocities, etc. Equations (6) and (7) permit many ways of defining the generalized speeds; one simple definition is $u_r = \dot{q}_j$, $r = j = 1,n$.

If Eq. (7) is substituted into Eq. (5) and terms are collected, the velocities of the particles can be written as

$$v^i = \sum_{r=1}^{n} v_r^i u_r + v_t^i, \qquad i = 1,n \tag{8}$$

Each vector v_r^i is called the rth partial velocity of the particle (or point) i, since

$$v_r^i = \frac{\partial v^i}{\partial u_r}, \qquad r = 1,n \tag{9}$$

Absolute angular velocities of rigid bodies would also be written in terms of the u_r and similarly defined partial angular velocities.

The vectors v_r^i and v_t^i are, in general, functions of the q_j and t; normally they are determined by inspection after performing the previously indicated substitutions. The components of v_t^i are distinguished by not being coefficients of the generalized speeds; these terms are generated by rheonomic constraints and explicit time-varying inputs. They are not used in forming Kane's equations and are usually discarded without any particular explanation.[1-3]

Equation (2) is now rewritten using Eqs. (5) and (8). On the left side (work terms) this gives

$$\sum_{i=1}^{p} F_i \cdot dr^i = \sum_{i=1}^{p} v^i \cdot F_i \, dt = \left(\sum_{r=1}^{n} \left[\sum_{i=1}^{p} v_r^i \cdot F_i \right] u_r \right) dt + \sum_{i=1}^{p} v_t^i \cdot F_i \, dt \tag{10}$$

Similarly, the right side of Eq. (2) becomes

$$\sum_{i=1}^{p} m_i a^i \cdot dr^i = \sum_{i=1}^{p} m_i a^i \cdot v^i \, dt = \left(\sum_{r=1}^{n} \left[\sum_{i=1}^{p} v_r^i \cdot (m_i a^i) \right] u_r \right) dt + \sum_{i=1}^{p} v_t^i \cdot (m_i a^i) \, dt \tag{11}$$

Equations (10) and (11) are equated, and terms are collected:

$$\left(\sum_{r=1}^{n} \left[\sum_{i=1}^{p} v_r^i \cdot F_i + \sum_{i=1}^{p} v_r^i \cdot (-m_i a^i) \right] u_r \right) dt + \left(\sum_{i=1}^{p} v_t^i \cdot (F_i - m_i a^i) \right) dt = 0 \tag{12}$$

The u_r, $r = 1,n$, and dt are nonzero and independent, and so the coefficients of each of these must be zero. Equation (12) has two groupings of terms: the first reflects system geometry ($u_r dt$) and the second reflects time (dt). The second sum is clearly zero since each term is zero by Eq. (1) and, therefore, provides no new information. In the first sum, we identify the generalized active forces and generalized inertia forces:

$$Q_r = \sum_{i=1}^{p} v_r^i \cdot F_i, \qquad r = 1,n \tag{13}$$

$$Q_r^* = \sum_{i=1}^{p} v_r^i \cdot (-m_i a^i), \qquad r = 1,n \tag{14}$$

Whence

$$Q_r + Q_r^* = 0, \qquad r = 1,n \tag{15}$$

Equations (13-15) comprise Kane's equations. They result solely from terms in the system geometry—the first sum in Eq. (12).

The specific form of the generalized active forces in Eq. (13) is very desirable: now, forces in the F_i that do no geometric work will not appear in the Q_r. These include the usual so-called nonworking (or ideal) forces of constraint: e.g., internal forces between particles or in inextensible members, reaction forces, forces of geometric constraint, etc., and, with reference to Eqs. (2) and (10), forces normal to or with no component in the feasible directions of motion at point i. Such forces must be included in F_i for Newton's equation; however, in forming Eq. (13), whether or not such forces are included in the analysis (in F_i), they will not appear in the generalized active forces. Any valid definition of the generalized speeds per Eqs. (6) and (7) yields these results.

If there are m nonholonomic constraints (linear relations among the $\dot{q}_j$ or the u_r), one can solve for $n - m$ independent generalized speeds, substitute these into Eq. (7), and proceed in the same manner with the derivation. The resulting equations are exactly the same as Eqs. (13-15), except that the number of Kane's equations is reduced from n to $n-m$, the actual number of degrees of freedom, in the remaining nonholonomic variables.

Thus, the exact forms of Kane's generalized forces—including the partial velocities—have been derived from the work-energy principle; rheonomic terms of the velocities in Eq. (8) can indeed be neglected in forming the generalized forces, as averred.[1-3]

Lagrange's Equations

Lagrange's equations can as easily be derived from the work-energy form, Eqs. (1-5), as from the virtual work principle. Virtual work-like terms become apparent in the results, albeit in a slightly different guise.

Equation (4) is substituted into Eq. (2) and the summations are reordered. On the right side of Eq. (2)

$$\sum_{i=1}^{p} m_i \ddot{r} \cdot \mathrm{d}r^i = \sum_{j=1}^{n}\left(\sum_{i=1}^{p} m_i \ddot{r}^i \cdot \frac{\partial r^i}{\partial q_j}\right)\mathrm{d}q_j + \left(\sum_{i=1}^{p} m_i \ddot{r}^i \cdot \frac{\partial r^i}{\partial t}\right)\mathrm{d}t \tag{16}$$

The coefficients of the $\mathrm{d}q_j$ can be written as

$$\sum_{i=1}^{p} m_i \ddot{r}^i \cdot \frac{\partial r^i}{\partial q_j} = \sum_{i=1}^{p}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{r}^i \cdot \frac{\partial r^i}{\partial q_j}\right) - m_i r^i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial r^i}{\partial q_j}\right)\right] \tag{17}$$

On the right side of Eq. (17), the following relations are used:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial r^i}{\partial q_j} = \frac{\partial \dot{r}^i}{\partial q_j} \tag{18}$$

$$\frac{\partial r^i}{\partial q_j} = \frac{\partial \dot{r}^i}{\partial \dot{q}_j} \tag{19}$$

$$T = \frac{1}{2}\sum_{i=1}^{p} m_i(\dot{r}^i \cdot \dot{r}^i) \tag{20}$$

Accordingly, the first term on the right side of Eq. (16) becomes

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{p} m_i \ddot{r}^i \cdot \frac{\partial r^i}{\partial q_j}\right)\mathrm{d}q_j = \sum_{j=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\mathrm{d}q_j \tag{21}$$

The left side of Eq. (2) is now written as

$$\sum_{i=1}^{p} F_i \cdot \mathrm{d}r^i = \sum_{j=1}^{n}\left(\sum_{i=1}^{p} F_i \cdot \frac{\partial r^i}{\partial q_j}\right)\mathrm{d}q_j + \left(\sum_{i=1}^{p} F_i \cdot \frac{\partial r^i}{\partial t}\right)\mathrm{d}t \tag{22}$$

In Eq. (22), the coefficients of the $\mathrm{d}q_j$ are defined as

$$Q_j = \sum_{i=1}^{p} F_i \cdot \frac{\partial r^i}{\partial q_j}, \qquad j = 1, n \tag{23}$$

Substitute Eq. (23) into Eq. (22), Eq. (21) into Eq. (16), equate the results per Eq. (2), and collect terms to get

$$\sum_{j=1}^{n}\left\{Q_j - \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\right\}\mathrm{d}q_j + \sum_{i=1}^{p}\left[(F_i - m_i \ddot{r}^i)\cdot\frac{\partial r^i}{\partial t}\right]\mathrm{d}t = 0 \tag{24}$$

All of the differentials in Eq. (24) are independent and nonzero. The second sum is from the rheonomic terms in Eq. (4). This sum is zero, since every term is zero by Eq. (1), and provides no new information. In the left-hand sum, the coefficient of each $\mathrm{d}q_j$ is equated to zero to yield n Lagrange's equations in the n generalized coordinates. The Lagrange generalized forces in Eq. (23) have the same properties as Kane's generalized forces: forces that do no geometric work are eliminated and consequently may or may not be included in forming the F_i.

Nonholonomic constraints are usually treated by appending them to T [Eq. (20)] (or the Lagrangian) with Lagrange multipliers; the n Lagrange equations and m constraints are then solved conjointly in the $n+m$ generalized coordinates and Lagrange multipliers. Alternately, one can postmultiply Lagrange's equations by the transformation from the $\dot{q}_j$ to the remaining nonholonomic variables.[1]

The terms that yield Lagrange's equations are the same as would occur using virtual work, and the manipulations are essentially the same as in those derivations.[5,7] However, here it has not been necessary to define virtual displacements and virtual work, nor to explain why $\mathrm{d}t$ is assumed to be zero, nor to establish the mathematics of differentiating virtual displacements. The additional work term resulting from retaining the rheonomic terms gives no new information on the system motions. These derivations suggest that the virtual work approach is a contrivance, or shortcut, and unnecessary for deriving Lagrange's equations.

Discussion

Clearly, these derivations of Kane's and Lagrange's equations are very similar and result in the same sets of terms. Starting from the work-energy form of Newton's second law, Eq. (2), both of the derivations led to an equation of the form

$$\sum_{j=1}^{n}([1])\,\mathrm{d}q_j + \sum_{i=1}^{p}([2])\,\mathrm{d}t = 0 \tag{25}$$

namely, Eqs. (12) and (24). The differentials $\mathrm{d}q_j$ (or equivalently $u_r\,\mathrm{d}t$) and $\mathrm{d}t$ are independent and nonzero. The first sum reflects the feasible motions of the system and the associated geometric work of all of the forces; this sum yields Kane's and Lagrange's equations and is analogous to the virtual work. The second sum comes from the rheonomic terms in Eq. (4) or (5) and merely returns Newton's second law for each particle; these terms are disregarded for Kane's equations and in defining virtual motions.

Kane's generalized active forces Q_r are related to Lagrange's generalized forces Q_j. The respective forces are identical if the generalized speeds are $u_r = \dot{q}_j$, $j = r = 1, n$. Then, Eq. (8) is identical to Eq. (5), and the partial velocities are

$$v_r^i = \frac{\partial r^i}{\partial q_r}, \qquad r = 1, n \tag{26}$$

In formulating a problem for Kane's or Lagrange's equations, F_i can include all constraint forces as well as the impressed forces, or constraint forces that do no geometric work can be omitted, since such forces are eliminated from the generalized active forces.

Kane's equations can directly incorporate nonholonomic constraints, reducing the number of equations and possibly simplifying their solution. In general, Lagrange's equations lead more easily to conservation principles. Kane's equations might be viewed as an intermediate form between Newton's and Lagrange's equations. Indeed, setting up Kane's equations for a problem essentially follows the Newton-Euler formulation.

The derivations clearly show the relationship of virtual work to the total work. Virtual work-type terms appear here as geometric work. The use of total differential motions does not affect the derivations and greatly reduces the amount of explanation needed. In applying the equations, the explicit time-varying terms are simply neglected: for Kane's equations, these are the v_t terms in Eq. (5); for Lagrange's equations, it is unnecessary to imagine virtual displacements. In both cases, only the geometrically possible motions are considered. The key step in both the derivation and implementation of these equations is the transformation to generalized coordinates reflecting these motions.

Summary

Kane's equations and Lagrange's equations have been derived from the work-energy form of Newton's second law—considering all the forces acting on the system and total differentials. This provides a formal derivation of Kane's generalized forces and the associated forms. The derivations clearly show the relationships between Lagrange's and Kane's equations and the virtual work principle. It was seen that the virtual work terms appeared quite naturally as geometric work, the preferred term here. Alternately, the virtual work/d'Alembert approach might be considered as derivable from the work-energy form of Newton's second law. Thus, creating and rationalizing virtual displacements and virtual work in the derivations and applications is unnecessary, and the actual computations are unaffected. The additional effort of using ordinary differentials and total work in the derivation is minimal, and it avoids a lot of cumbersome explanations.

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Stabilizing Control for Second-Order Models and Positive Real Systems

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Introduction

RECENT activity in the control of large flexible space structures has led to an increasing algorithmic interest in the second-order linear models by which they are often approximated. This Note examines some of the algebraic properties of these models with the goal of developing efficient computational algorithms for their analysis and control. The main result appears as Theorem 7, which provides an easily computed stabilizing control law for second-order models. This theorem is more general than results that have appeared previously[1,2] in that it allows for a singular stiffness matrix, corresponding to rigid-body modes, and nonsymmetric damping matrix.

These results are also related to positive real systems. It is shown that, for a positive real system with input vector $\boldsymbol{u}$ and output vector $\boldsymbol{y}$, the control $\boldsymbol{u} = -R^{-1}\boldsymbol{y}$, where R is any positive definite matrix, is not only stabilizing but optimal in a linear-quadratic sense for a particular choice of weighting matrices.

Definitions

The generic form of what we shall call a second-order model is a system of differential equations

$$M\ddot{x} + C\dot{x} + Kx = Bu \tag{1}$$

with an associated output equation

$$y = H_1 x + H_2 \dot{x} \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, $y = \mathbb{R}^r$, and where the model matrices are assumed to have the following structure:

$$M = M^T > 0 \qquad K = K^T \geq 0$$

$$C = C_1 + C_2; \qquad C_1^T = C_1 \geq 0, \qquad C_2^T = -C_2$$

In this model, M is the mass matrix, C_1 represents structural damping forces, C_2 is the gyroscopic matrix, and K is the stiffness matrix. This model ignores any so-called circulatory forces, which would appear as a skew-symmetric matrix added to K. For further discussion of the significance and limitations of this model, see Arnold,[3] Balas,[4] and Meirovitch.[5]

A second-order model is usually converted to an equivalent first-order form before any control computations are performed. Its special structure is then ignored as standard computational algorithms are used. There are several ways to express Eq. (1) as an equivalent first-order model. One way of doing this is

$$E\dot{z} = Az + Lu$$

$$y = Hz$$

where

$$z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix}, \qquad E = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}$$

$$L = \begin{pmatrix} 0 \\ B \end{pmatrix}, \qquad H = (H_1 \quad H_2)$$

The eigenvalue problem associated with Eq. (1) is

$$[s^2M + sC + K]x = 0 \tag{3}$$

where the eigenvalue s is a complex scalar and the eigenvector $\boldsymbol{x}$ is a complex n vector.

Properties of Second-Order Models

Surprisingly few results are available concerning the properties of second-order models from the computational point of view. There are a few exceptions. Meirovitch[5] discusses the eigenstructure of the first-order form for certain special cases of the model. Shieh et al.[6] have published a collection of results on the dynamic stability of systems represented by second-order models. Many results on lambda matrices, of which Eq. (3) is a special case, appear in Lancaster.[7] Laub and Arnold[8] and Bender and Laub[9] have developed tests for con-

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